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# A PROOF OF REIDEMEISTER-SINGER'S THEOREM BY CERF'S METHODS

FRANÇOIS LAUDENBACH

**ABSTRACT.** Heegaard splittings and Heegaard diagrams of a closed 3-manifold  $M$  are translated into the language of Morse functions with Morse-Smale pseudo-gradients defined on  $M$ . We make use in a very simple setting of techniques developed by Jean Cerf for solving a pseudo-isotopy problem.

## 1. INTRODUCTION

When speaking of Cerf's methods, or Cerf theory, we refer to the work of Cerf in [1] for the so-called *pseudo-isotopy* problem. Briefly said, the method consists of reducing some isotopy problem to a problem about real functions. It was created in the setting of high dimensional manifolds. But some parts apply in dimension three. The purpose of this note is to present a proof of Reidemeister-Singer's theorem (as stated below) in this way. I should say that Francis Bonahon, who like me was educated in the Orsay Topology group of the seventies-eighties, wrote such a proof; but, his notes are lost. The recent developments in Heegaard-Floer homology drove me to make this proof available.

**Theorem 1.1. (Reidemeister [6, 7], Singer[8])** *Let  $M$  be a closed 3-manifold.*

- 1) *Two Heegaard splittings become isotopic after suitable stabilisations.*
- 2) *More precisely, let  $D_0, D_1$  be two Heegaard diagrams. Then, there are stabilisations  $D'_0, D'_1$  by adding pairs of cancelling handles of index 1 and 2, such that one passes from  $D'_0$  to  $D'_1$  by an ambient isotopy and a finite sequence of handle slidings.*

Strictly speaking, only the first item is the statement of the Reidemeister-Singer theorem; the reader is referred to the final comments at the end of this note. A *Heegaard splitting* consists of a closed surface  $\Sigma$  of genus  $g$ , called *Heegaard surface*, dividing  $M$  into two handlebodies  $H^-, H^+$ . A *Heegaard diagram* is defined by a more precise data, namely, a handle decomposition of  $M$  with:

- one 0-handle  $B^-$  and  $g$  handles of index 1 attached on the boundary  $\partial B^-$ , whose union forms  $H^-$ ;
- $g$  handles of index 2 attached on  $\partial H^-$  and one 3-cell  $B^+$  whose union forms  $H^+$ .

On the common boundary  $\Sigma$  of  $H^+$  and  $H^-$ , the Heegaard diagram specifies  $g$  simple curves  $\beta_1, \dots, \beta_g$  in  $\Sigma$ , mutually disjoint, which are the cores of the attaching domains of the 2-handles;

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their complement in  $\Sigma$  is a 2-sphere with  $2g$  holes. It also specifies  $g$  simple curves  $\alpha_1, \dots, \alpha_g$  which are the boundaries of the so-called *transverse* 2-cells of each 1-handle (with a terminology slightly different from [5], Section 3); the complementary in  $\Sigma$  of  $\cup_j \alpha_j$  is also a 2-sphere with  $2g$  holes.

The statement of theorem 1.1 can be translated into the language of Morse functions as follows. Recall that a Morse function is said to be *ordered* if the order of the critical values is finer than the order of their indices. In dimension 3, an ordered Morse function gives rise to a Heegaard splitting by considering a level set whose level separates the index 1 and index 2 critical values. Moreover, every Heegaard splitting is obtained this way. Along a path of ordered Morse functions one has a Heegaard surface moving by isotopy.

Stabilisation of a Heegaard splitting consists of creating a pair of critical points of index 1 and 2 at a level keeping the ordering. Thus, item 1 of theorem 1.1 is a consequence of the next theorem which holds true in every dimension  $\geq 2$ .

It is useful to recall that, in a generic path of smooth functions  $(f_t)_{t \in [0,1]}$ , for all  $t \in [0,1]$  but a finite number of  $t_j$ 's,  $f_t$  is Morse and  $f_{t_j}$  has non-degenerate critical points but one whose Hessian has corank 1; moreover  $t_j$  is a *birth time* or a *cancellation time*, meaning that  $f_{t_j+\varepsilon}$  has one more or one less pair of critical points than  $f_{t_j-\varepsilon}$ . In this note, all genericity arguments follow from Thom's *transversality theorem in jet spaces* as it is done in his article on singularities [9] (see also [3], or [4] - in French - where the generic paths of real functions are explicitly considered).

**Theorem 1.2.** *Let  $M$  be a closed manifold of dimension  $\geq 2$ . Given two ordered Morse functions  $f_0, f_1$  on  $M$ , there exists a generic path of functions  $(f_t)_{t \in [0,1]}$  such that, for every  $t \in [0,1]$  but a finite set  $J = \{t_1, \dots, t_q, t_{q+1}, \dots, t_{q+q'}\}$ ,  $f_t$  is an ordered Morse function. Moreover,  $t_1, \dots, t_q$  are birth times and lie in  $(0, 1/3)$ ; and  $t_{q+1}, \dots, t_{q+q'}$  are cancellation (or death) times and lie in  $(2/3, 1)$ .*

*In particular in dimension 3, a level set of  $f_{1/2}$  whose level separates the index 1 and index 2 critical values is a Heegaard splitting which is a common stabilisation of those associated to  $f_0$  and  $f_1$ .*

In order to speak of handle decomposition and handle sliding, it is useful to consider a Morse function  $f$  equipped with a *pseudo-gradient*. It is a smooth vector field  $X$  on  $M$  which satisfies the Lyapunov inequality  $X \cdot f < 0$  outside the critical locus and some non-degeneracy condition at each critical point  $p$ : the Hessian at  $p$  of  $X \cdot f$  is negative definite (notice that  $X \cdot f \leq 0$  everywhere). Local data of pseudo-gradients generate a global pseudo-gradient by partition of unity.

For each critical point  $p$  of  $f$ , there are stable and unstable manifolds denoted respectively by  $W^s(p, X)$  and  $W^u(p, X)$ . Generically,  $X$  is *Morse-Smale*, meaning that all the invariant manifolds of critical points are mutually transverse. An ordered Morse function  $f$  with a Morse-Smale pseudo-gradient  $X$  gives rise easily to a handle decomposition.

But, given a path  $(f_t)_{t \in [0,1]}$  of ordered Morse functions and a generic path  $(X_t)_{t \in [0,1]}$ , where  $X_t$  is a pseudo-gradient of  $f_t$  for every  $t \in [0,1]$ , there exists a finite set  $K = \{t_1, \dots, t_r\}$  such that, for each  $t_k \in K$ , exactly one  $X_{t_k}$ -orbit  $\ell_k$  connects two critical points  $p$  and  $p'$  having the

same index; moreover, for each  $x \in \ell_k$ , we have:

$$T_x \ell_k = T_x W^u(p, X_{t_k}) \cap T_x W^s(p', X_{t_k}),$$

and  $t \mapsto X_t$  crosses transversely at time  $t_k$  the codimension-one stratum of the space of pseudo-gradients having a connecting orbit between two critical points with the same index. One says that a *handle sliding* happens at time  $t_k$ . The effect of a handle sliding on the so-called *Morse complex* is described by J. Milnor in [5], Theorem 7.6. It is useful to emphasize that, in a generic path  $(f_t, X_t)_t$  and when  $j > i$ , it never appears an  $i/j$  connecting orbit, that is, an orbit descending from a critical point of index  $i$  to a critical point of index  $j$ .

Now, the statement of item 2) in theorem 1.1 can be translated into the next one. For simplicity, a function with two local extrema will be said to be *simple*.

**Theorem 1.3.** *Let  $M$  be a closed manifold of dimension  $n > 2$ . Given two ordered simple Morse functions  $f_0, f_1$  equipped with respective Morse-Smale pseudo-gradients  $X_0, X_1$ , there exists a generic path of pairs  $(f_t, X_t)_{t \in [0,1]}$  such that, for every  $t \in [0, 1]$  but a finite set  $f_t$  is an ordered simple Morse function and  $X_t$  is a Morse-Smale pseudo-gradient of  $f_t$ . The excluded  $t$ 's are times of birth, cancellation or handle sliding.*

The main lemma concerns reordering, that is, a process which is efficient for crossing critical values. Here it is.

**Lemma 1.4.** *Let  $f : M \rightarrow \mathbb{R}$  be a Morse function,  $X$  be a pseudo-gradient of  $f$ , and  $p$  be a critical point. Assume that the unstable manifold  $W^u(p, X)$  contains a closed smooth  $k$ -disc  $D$ ,  $k = \text{index}(p)$ , whose boundary lies in a level set  $f^{-1}(a)$ ,  $a < f(p)$ . Then, for every  $\varepsilon > 0$ , there exists a path  $(f_t)_{t \in [0,1]}$  of Morse functions such that  $f_0 = f$ ,  $f_1(p) = a + \varepsilon$  and  $X$  is a pseudo-gradient of  $f_t$  for every  $t \in [0, 1]$ . Moreover the support of the deformation may be contained in an arbitrarily small neighborhood of  $D$ .*

The lemma above holds true in a family (same proof). Moreover,  $f$  has only to be a Morse function in a neighborhood of  $D$ . In particular, it applies for a generic path of functions.

## 2. PROOFS

**2.1. Proof of lemma 1.4.** Set  $c = f(p)$  and assume that  $\varepsilon$  is small enough so that the interval  $[c, c + \varepsilon]$  contains no critical value. Let  $U$  be a tubular neighborhood of  $\partial D$  in  $f^{-1}(a)$  and let us consider the union  $\mathcal{M}$  of  $D$  and all segments of  $X$ -orbits starting from points in  $f^{-1}(c + \varepsilon)$  and ending at  $p$  or in  $U$ . It is somehow similar to a Morse model for the pseudo-gradient. Its boundary is made of three parts, two horizontal parts  $\mathcal{M} \cap f^{-1}(a)$  and  $\mathcal{M} \cap f^{-1}(c + \varepsilon)$ , and the lateral boundary  $\partial_\ell \mathcal{M}$  which is tangent to  $X$ . There are two corners in the boundary of  $\mathcal{M}$ , each being diffeomorphic to a product of spheres  $S^{k-1} \times S^{n-k-1}$  (where  $k = \text{index}(p)$ ); one is the boundary of  $U$  trivialized as the sphere bundle  $\partial U \rightarrow \partial D$ ; the other corner is  $\partial_\ell \mathcal{M} \cap f^{-1}(c + \varepsilon)$  and is diffeomorphic to the first one by the flow of  $X$ .

Let  $N$  be a small collar neighborhood of  $\partial_\ell \mathcal{M}$  in  $\mathcal{M}$ ; it is diffeomorphic to a product

$$N \cong S^{k-1} \times S^{n-k-1} \times [0, 1]^2,$$

where the product structure is chosen so that the vertical lines of the square are tangent to the orbits and the horizontal lines are contained in level sets of  $f$ .

The trick consists of replacing the horizontal foliation of the square with a new one which is still transverse to the vertical foliation, still horizontal near the boundary, and puts  $f^{-1}(a + \varepsilon) \cap \{x = 0\}$  on the same leaf as  $f^{-1}(c) \cap \{x = 1\}$ , where  $x$  denotes the first coordinates of the square (see figure 1).

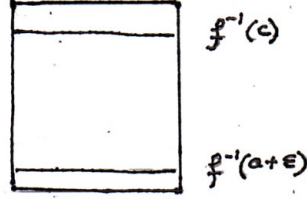


Figure 1A

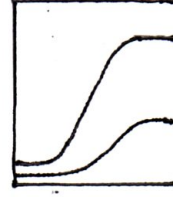


Figure 1B

□

**Corollary 2.2.** *Let  $(f_o, X)$  be a Morse function with a pseudo-gradient with no  $i/j$  connecting orbit,  $j > i$ . Then there exists a path  $(f_t)_{t \in [0,1]}$  of Morse functions such that  $f_1$  is ordered and  $X$  is a pseudo-gradient of  $f_t$  for every  $t \in [0, 1]$ .*

It is even possible to have  $f_1$  *self-indexing*, meaning that  $f(p) = \text{index}(p)$ .

**Proof.** For brevity, the proof is given in dimension 3 only. We are free to decrease arbitrarily the value of the minima and to increase the value of the maxima. Then, in dimension 3, the matter is to put the critical values of index 1 below those of index 2. If the function is not ordered, there is a pair of critical points  $(p, q)$  with  $p$  of index 1,  $q$  of index 2 and  $f(p) \geq f(q)$ . Choose such a pair so that  $f(p)$  and  $f(q)$  are minimal among such *unordered pairs*. By this choice  $W^u(p, X)$  has two separatrices crossing a level set below  $f(q)$ ; if not, one separatrix of  $p$  ends at a critical point  $p'$  of index 1 with  $f(p) > f(p') \geq f(q)$ , contradicting the assumption on the pair  $(p, q)$ . Then, lemma 1.4 applies and yields a new Morse function which has the same pseudo-gradient  $X$  and at least one unordered pair less than  $f$ . Arguing this way recursively, the corollary is proven. □

Before proving theorem 1.2, it is useful to specify what a *birth path* is.

**Definition 2.3.** *A birth path is a generic path of functions  $(f_t)_{t \in [0, \varepsilon]}$  defined on an  $n$ -dimensional manifold  $M$  such that there exists a path of cylinders  $B_t \cong D^{n-1} \times [-1, +1]$  embedded in  $M$  with the following properties for every  $t \in [0, \varepsilon]$ :*

- $D^{n-1} \times \{\pm 1\}$  lies in two level sets of  $f_t$ ;
- the restriction of  $f_t$  to  $\partial D^{n-1} \times [-1, +1]$  has no critical points;
- $f_t|_{B_t}$  is conjugate to the function  $F_t(x, y) := x^3 - (t - \frac{\varepsilon}{2})x + q(y)$ , restricted to some cylinder  $\mathcal{B}_t \subset \mathbb{R} \times \mathbb{R}^{n-1}$  centered at the origin, where  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}^{n-1}$  and  $q$  is a non-degenerate quadratic form.

At time  $t = 0$ , the function  $f_0$  has no critical points in  $B_0$  while, at time  $t = \varepsilon$ , the function  $f_\varepsilon$  has a pair of critical points in  $B_\varepsilon$  of respective index  $i, i + 1$  if  $q$  has index  $i$ . Notice that, if  $f_0$  is given with a cylinder  $B_0$  on which  $f_0$  induces the height function, then  $f_0$  is the starting point of a birth path supported in  $B_0$ . The same remark in a 1-parameter family yields the next elementary lemma.

**Lemma 2.4.** *Let  $(h_s)_{s \in [0,1]}$  be a generic path of functions on  $M$  and  $(\beta_t^1)$  be a birth path starting from  $h_1$  with associated cylinders  $B_t^1$ . Then, there is a family (parametrized by  $s \in [0, 1]$ ) of birth paths  $\beta_t^s$  associated to cylinders  $B_t^s$  and starting from  $h_s$ . Moreover, if  $M$  is connected, it is possible to choose the  $B_t^0$ 's centered at any regular point of  $h_0$ .*

*As a consequence the composed path  $(h_s) * (\beta_t^1)$  is homotopic, relative to its end points, to the path  $(\beta_t^0) * (\beta_1^s)$ .*

In other words, in a generic path, a time of birth can be shifted to the left of the time interval and the place of the birth can be located in a small neighborhood of any regular point. By reversing the time, any cancellation time can be shifted to the right of the time interval, etc.

**2.5. Proof of theorem 1.2.** Given two ordered Morse functions  $f_0, f_1$ , there exists a generic path  $(f_t)_{t \in [0,1]}$  where  $f_t$  is Morse for every  $t \in [0, 1]$  but a finite set  $J$ . Decompose  $J = J_+ \cup J_-$  where  $J_\pm$  is the set of birth/cancellation times and apply lemma 2.4. The birth times  $J_+$  can be shifted to the left, say in  $[0, t_0]$ , and the cylinders of birth can be located so that all Morse functions in  $[0, t_0]$  are ordered. Similarly, the cancellation times can be shifted to the right, say in  $[t_1, 1]$ , and the cancellation cylinders can be chosen so that all Morse functions in  $[t_1, 1]$  are ordered. Thus,  $f_t$  is a Morse function for every  $t \in [t_0, t_1]$ .

Choose vector fields  $X_t$  depending smoothly on  $t \in [t_0, t_1]$  which are pseudo-gradients of  $f_t$ . As we said before stating theorem 1.3, by taking  $(X_t)_{t \in [t_0, t_1]}$  generic,  $X_t$  is Morse-Smale for all  $t \in [0, 1]$  but a finite set  $K \subset (t_0, t_1)$ .

Apply corollary 2.2 to the functions  $f_{t_k}$ ,  $t_k \in K$ , and deform the path of functions accordingly, keeping the same  $X_t$ 's as pseudo-gradients. After that deformation, the functions  $f_{t_k}$ ,  $t_k \in K$ , are ordered and, for every  $t \in (t_k, t_{k+1})$ , the vector fields  $X_t$  is Morse-Smale (no handle sliding); it is also true for every  $t \in (t_0, t_1)$  on the left or right of  $K$ . So, we are reduced to reorder a path of Morse functions equipped with pseudo-gradients which are Morse-Smale for every time. The reordering is then obtained by applying the one-parameter version of lemma 1.4.  $\square$

This finishes the proof of item 1) in theorem 1.1. Before proving theorem 1.3 and, hence, item 2) in theorem 1.1, we need to recall the *swallow tail lemma*, which is similar to the *cancellation lemma* of Smale (see J. Milnor [5], Section 5). We state it by means of Cerf graphics. Recall that the *Cerf graphic* of a path of functions  $(h_t)_t$  is the part of  $\mathbb{R}^2$  whose intersection with  $\{t\} \times \mathbb{R}$  is the set of critical values of  $h_t$ .

**Lemma 2.6. (Swallow tail lemma).** *Let  $\gamma := (f_t)_{t \in [0,1]}$  be a generic path whose Morse functions are ordered. Assume that its restriction to  $t \in [t_0, t_1]$  has a Cerf graphic as on figure 2A: there are three critical points  $p_t, p'_t$  of index  $i+1$  and  $q_t$  of index  $i$  such that the pair  $(p_t, q_t)$  is created at time  $t_0$  and the pair  $(p'_t, q_t)$  is cancelled at time  $t_1$ ; moreover, there are no other critical values in  $[f_t(q_t), \max\{f_t(p_t), f_t(p'_t)\}]$ . Then  $\gamma$  can be deformed relative to  $[0, t_0 - \varepsilon] \cup [t_1 + \varepsilon, 1]$  until a path  $\gamma'$  whose Cerf graphic is trivial over  $[t_0, t_1]$  as on figure 2B.*

A similar result holds true for the upside down Cerf graphic where  $q_t$  has index  $i + 1$  and  $p_t, p'_t$  have index  $i$ .

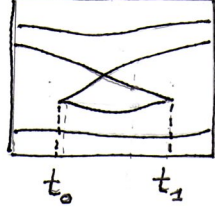


Figure 2A

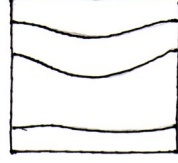


Figure 2B

In what follows, we shall use this lemma in the case  $i = 0$  only. The assumption on the critical values implies that there is a unique connecting orbit from  $p_t$  to  $q_t$  as long as  $f_t(p_t) \leq f_t(p'_t)$  and a unique connecting orbit from  $p'_t$  to  $q_t$  when  $f_t(p_t) \geq f_t(p'_t)$  (for any pseudo-gradient). Indeed, for changing the number of connecting orbits from  $p_t$  to  $q_t$ , it is necessary to make a handle sliding and then, to have an auxiliary critical point whose value lies in  $(f_t(q_t), f_t(p_t))$ ; the form of the graphic prevents us to be in such a situation.

In the above lemma, the deformation passes through a function with one singularity of codimension 2, which reads in coordinates  $\pm x^4 + q(y)$ , where  $q(y)$  is a non-degenerate quadratic form on  $\mathbb{R}^{n-1}$ .

The swallow tail lemma is clear when the dimension of  $M$  is 1. When  $\dim M > 1$  and  $i = 0$ , there are coordinates  $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$  in a neighborhood  $U$  of the 1-dimensional separatrices of critical points in question such that  $f_t(x, y) = h_t(x) + q(y)$  where  $(h_t)_t$  has the same Cerf graphic as  $(f_t)_t$  and  $q(y)$  in a quadratic form positive definite (for the general case, see Cerf [1], Chap. IV §4). Of course, it is necessary to control the support of the deformation and insure that no extra critical point appears. For instance, the true formula of deformation reads

$$s \mapsto h_t(x) + \omega(|y|)k_{s,t}(x) + q(y),$$

where  $\omega$  is a bump function with compact support centered at 0 and  $k_{s,t}(x)$  is non-positive. Then, the  $y$ -derivative vanishes at  $y = 0$  only, and the critical points are those of the one-dimensional case.

**2.7. Proof of theorem 1.3.** According to theorem 1.2, there is a path  $\gamma := (f_t)_t$  fulfilling all requirements of theorem 1.3 (birth times before cancellation times and order of critical values) but the one min/one max condition. So, the matter is to kill the appearance of extra local minima or maxima. We are looking at the local minima only. There are two cases.

**CASE 1:** One can follow continuously a minimum  $m_t$  of  $f_t$  from  $t = 0$  to  $t = 1$ . Using that unnecessary crossings of index 0 critical values can be avoided, we may assume that the index 0 part of the Cerf graphic shows no crossings (see figure 3A). Let  $\mu$  be the maximal number of extra minima along  $\gamma$ ; we are going to decrease  $\mu$  by 1. Denote  $[t'_0, t'_1]$  the interval where  $f_t$  has  $\mu$  extra minima. For  $t \in [t'_0, t'_1]$ , denote the upper local minimum of  $f_t$  by  $m'_t$ .

Assume that  $3/2$  separates the index 1 critical values from those of index 2 and set  $L_t := f_t^{-1}(3/2)$ . If  $X_t$  is a pseudo-gradient of  $f_t$ , we see in  $L_t$  the trace  $B_t$  of the stable manifold  $W^s(m_t, X_t)$  and, when  $t \in [t'_0, t'_1]$ , the trace  $B'_t$  of the stable manifold  $W^s(m'_t, X_t)$ . Both are

changing when handle slidings of index 1 happen. But, due to  $n \geq 3$ , they remain connected; indeed, each one is always an  $(n - 1)$ -sphere with holes.

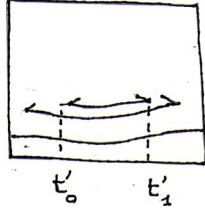


Figure 3A

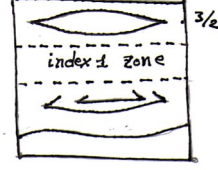


Figure 3B

So, choose continuously points  $x_t \in B_t$  and  $x'_t \in B'_t$  linked by a simple arc  $\alpha_t$  in  $L_t$ . We introduce a cancelling pair of critical  $(s_t, r_t)$  of respective index  $(2, 1)$  in a collar neighborhood above  $L_t$ ; the birth time is less than  $t'_0$ , the cancelling time greater than  $t'_1$  (compare figure 3B), and the base of the birth cylinder is a  $(n - 1)$ -disc in  $L_t$  centered at  $x_t$ . Denote by  $\gamma' := (f'_t)_t$  this new path from  $f_0$  to  $f_1$ . After choosing a suitable pseudo-gradient  $X'_t$ , we have for every  $t \in [t'_0 + \varepsilon, t'_1 - \varepsilon]$ :

$$W^u(r_t, X'_t) \cap L_t = \{x_t, x'_t\}, \quad W^u(s_t, X'_t) \cap L_t = \alpha_t.$$

In particular, there are no  $X'_t$ -connecting orbits from  $r_t$  to another critical point of index 1. Therefore, lemma 1.4 applies and a new deformation of the path  $\gamma'$  puts the critical value of  $r_t$  below the other critical values of index 1 when  $t \in [t'_0 + 2\varepsilon, t'_1 - 2\varepsilon]$  (compare the Cerf graphic on figure 4A). By the choice of  $x'_t$ , there is exactly one connecting orbit from  $r_t$  to  $m'_t$ . We choose some  $\theta \in [t'_0 + 2\varepsilon, t'_1 - 2\varepsilon]$  and cancel the pair  $(r_\theta, m'_\theta)$  from the function  $f'_\theta$ , according to Smale's classical lemma (*ibid.*). This cancellation may be viewed as a new deformation of the path  $\gamma'$ ; the final Cerf graphic looks like figure 4B, with two swallow tails. Lemma 2.6 applies and yields some deformation of the path of functions so that the swallow tails vanish. The final path of this last deformation has  $\mu - 1$  extra minima.

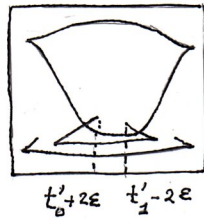


Figure 4A

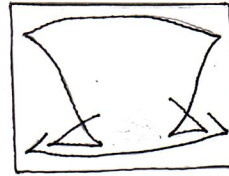


Figure 4B

CASE 2. The unique minimum  $m_0$  of  $f_0$  does not survive until time 1. It can be followed as  $m_t$ , an index 0 critical point of  $f_t$ , and it cancels at time  $t_1 < 1$  with an index 1 critical point  $s_t$ , well defined for  $t$  close to  $t_1$ ,  $t < t_1$ . In the same way, the minimum  $m'_1$  of  $f_1$  can be followed backward as  $m'_t$ , an index 0 critical point of  $f_t$  for  $t \in ]t_0, 1]$ . It is born at time  $t_0$  with an index 1 critical point  $s'_t$ . By using lemma 2.4 for shifting some birth times to the left and some cancellation times to the right, we may assume that  $t_0$  is the last birth time and  $t_1$  is the first cancellation time. So, the Cerf graphic of our path from  $f_0$  to  $f_1$  looks as shown on figure 5A, as far as the critical values of index 0 are concerned. The time where the values of  $m_t$  and  $m'_t$  are crossing is denoted by  $\tau$ .

Since  $M$  is connected, there exist a Morse-Smale pseudo-gradient  $X_\tau$  of  $f_\tau$  and an index 1 critical point  $p_\tau$  whose unstable manifold  $W^u(p_\tau, X_\tau)$  has one separatrix ending at  $m_\tau$  and the other at  $m'_\tau$ . According to lemma 1.4, we may assume that, for  $t$  close to  $\tau$ , the value of  $p_t$  is the lowest index 1 critical value. Therefore, the pair  $(p_t, m'_t)$  is in position of cancellation for  $t \in [\tau - \varepsilon, \tau]$  and the pair  $(p_t, m_t)$  is in position of cancellation for  $t \in [\tau, \tau + \varepsilon]$ . Then, at time  $\tau - \varepsilon$ , one achieves the cancellation of  $p_{\tau-\varepsilon}$  with  $m'_{\tau-\varepsilon}$ , and, at time  $\tau + \varepsilon$ , one achieves the cancellation of  $p_{\tau+\varepsilon}$  with  $m_{\tau+\varepsilon}$ . The new Cerf graphic is shown on figure 5B. It presents an upsided down swallow tail which can be erased thanks to lemma 2.6. After this deformation there is a minimum of  $f_t$  which can be followed from time 0 until time 1 and so we are reduced to case 1. Nevertheless, this process creates birth times greater than cancellation times. This can be repaired by applying lemma 2.4.  $\square$

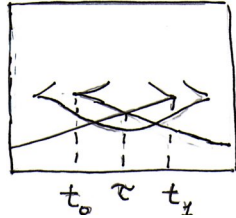


Figure 5A

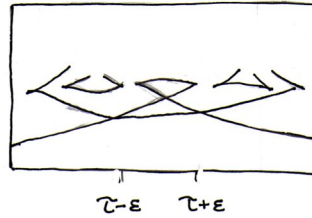


Figure 5B

## 2.8. Final comments.

1) According to R. Craggs [2], the Reidemeister-Singer theorem, that is, item 1 in theorem 1.1, was not considered as completely proven before his own 1976 short note.

2) It is worth noticing that both parts of theorem 1.1 are consequence of two statements (theorems 1.2 and 1.3) about functions which hold true in any dimension ( $>2$  for theorem 1.3). These two theorems should be known to specialists. Maybe, the proof of theorem 1.2 that is given here is almost the simplest one. I did not find any written proof of theorem 1.3.

3) The proof of the latter theorem is not very elementary, due to the way of using the swallow tail lemma. So, the 3-dimensional proof of item 2 in theorem 1.1 remains competitive. The statement reads as this: *Let  $H$  be a 3-dimensional handlebody of genus  $g$ , and let  $\mathcal{D}, \mathcal{D}'$  be two systems of  $g$  compression discs of  $H$  whose complements are connected. Then, one passes from  $\mathcal{D}$  to  $\mathcal{D}'$  by finitely many handle slidings.* This can be proven by the very standard *cut-and-past* technique.

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LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UMR 6629 DU CNRS, FACULTÉ DES SCIENCES ET TECHNIQUES, UNIVERSITÉ DE NANTES, 2, RUE DE LA HOUSSINIÈRE, F-44322 NANTES CEDEX 3, FRANCE.  
*E-mail address:* `francois.laudenbach@univ-nantes.fr`